Gov 2000 - 9. Multiple Linear Regression: Interactions and Nonlinearities

Matthew Blackwell
Harvard University
mblackwell@gov.harvard.edu

Where are we? Where are we going?

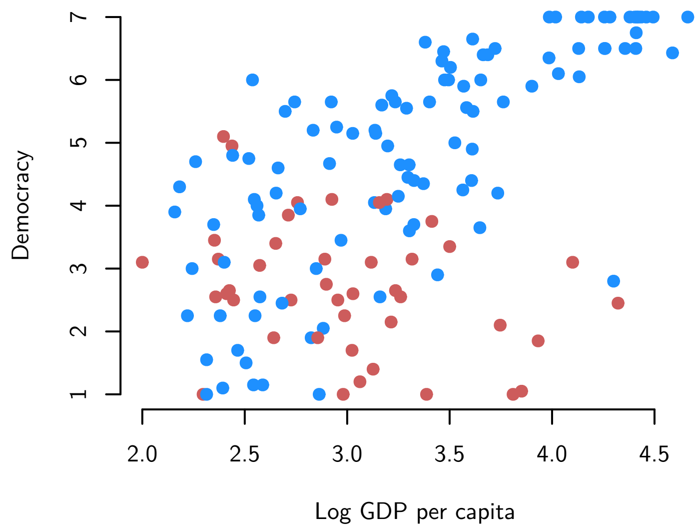
- Last few weeks: linear regression at its most general, matrix form
- This week: effects that vary between groups and other loose ends
- Next week: troubleshooting the linear model

INTERACTIONS

Data

- Data comes from Fish (2002), "Islam and Authoritarianism."
- Basic relationship: does more economic development lead to more democracy?
- We measure economic development with log GDP per capita
- We measure democracy with a Freedom House score, 1 (less free) to 7 (more free)

```
load("FishData.RData")
plot(FishData$income, FishData$fhrev, ylab = "Democracy", xlab = "Log GDP per capita",
    pch = 19, bty = "n", col = ifelse(FishData$muslim == 1, "indianred", "dodgerblue"))
```



- We might want to control for whether or not the country's largest religion is Islam.
- Why? Fish argues that Muslim countries are less likely to be democratic no matter their economic development.
- Let's put this to data and control for a binary variable `muslim` that is 1 when Islam is the largest religion in a country and 0 otherwise:

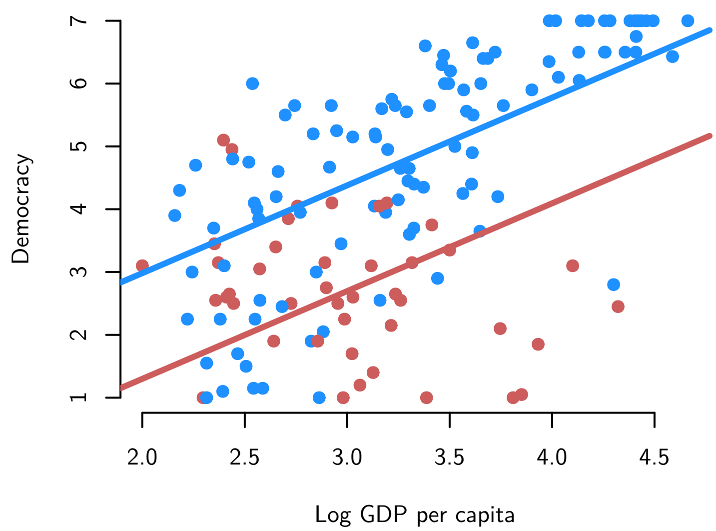
```
mod <- lm(fhrev ~ income + muslim, data = FishData)
summary(mod)
```

```
##
## Call:
## lm(formula = fhrev ~ income + muslim, data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.3961 -0.8276  0.2804  0.9425  3.2467
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.1887     0.5560   0.339    0.735
## income        1.3970     0.1629   8.576 1.31e-14 ***
## muslim       -1.6827     0.2379  -7.074 5.82e-11 ***
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.282 on 146 degrees of freedom
## Multiple R-squared:  0.5216, Adjusted R-squared:  0.515
## F-statistic: 79.58 on 2 and 146 DF,  p-value: < 2.2e-16
```

- Since `muslim` here is a binary variable, we can plot the two parallel regression lines implied by this model:

```
plot(FishData$income, FishData$fhrev, ylab = "Democracy", xlab = "Log GDP per capita",
    pch = 19, bty = "n", col = ifelse(FishData$muslim == 1, "indianred", "dodgerblue"))
abline(a = coef(mod)[1], b = coef(mod)[2], col = "dodgerblue", lwd = 3)
abline(a = coef(mod)[1] + coef(mod)[3], b = coef(mod)[2], col = "indianred",
    lwd = 3)
```



- But looking at the data here, we might notice that the red line for Muslim countries does not fit the lines very well. Maybe there is a different relationship between income and democracy in Muslim and non-Muslim countries.

Interaction between binary and continuous variables

- Let Z_i be binary
- In this case, $Z_i = 1$ for the country being Muslim

- We can add another covariate to the baseline model that allows the effect of income to vary by Muslim status.
- This covariate is called an interaction term and it is the product of the two **marginal** variables of interest:

$$income_i \times mulsim_i$$

- Here is the model with the interaction term:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- Literally this last term is just a new covariate that is the X_i multiplied by Z_i.

Example of binary interaction terms

- In R, we simply add a new term to the regression which is `first:second` where `first` and `second` are the names of marginal variables:

```
mod.int <- lm(fhrev ~ income + muslim + income:muslim, data = FishData)
summary(mod.int)
```

```
##
## Call:
## lm(formula = fhrev ~ income + muslim + income:muslim, data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.8460 -0.5705  0.0940  0.8517  2.6307
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -1.3489     0.5400  -2.498   0.0136 *
## income          1.8592     0.1590  11.695  < 2e-16 ***
## muslim          5.7413     1.1338   5.064 1.23e-06 ***
## income:muslim  -2.4267     0.3642  -6.662 5.23e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.125 on 145 degrees of freedom
## Multiple R-squared:  0.6337, Adjusted R-squared:  0.6261
## F-statistic: 83.61 on 3 and 145 DF,  p-value: < 2.2e-16
```

- Let's look at the design matrix to see what this looks like:

```
head(model.matrix(mod.int))
```

```
##   (Intercept)   income muslim income:muslim
## 1           1 2.925312      1      2.925312
## 2           1 3.214314      1      3.214314
## 3           1 2.824126      0      0.000000
## 4           1 3.762078      0      0.000000
## 5           1 3.187803      0      0.000000
## 6           1 4.435542      0      0.000000
```

- Note that it is easier and better to write the interaction term as `first*second`, which adds each variable and its interaction to the model:

```
mod.int <- lm(fhrev ~ income * muslim, data = FishData)
summary(mod.int)
```

```
##
## Call:
## lm(formula = fhrev ~ income * muslim, data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.8460 -0.5705  0.0940  0.8517  2.6307
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -1.3489     0.5400  -2.498   0.0136 *
## income          1.8592     0.1590  11.695  < 2e-16 ***
## muslim          5.7413     1.1338   5.064 1.23e-06 ***
## income:muslim  -2.4267     0.3642  -6.662 5.23e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.125 on 145 degrees of freedom
## Multiple R-squared:  0.6337, Adjusted R-squared:  0.6261
## F-statistic: 83.61 on 3 and 145 DF,  p-value: < 2.2e-16
```

Two lines in one regression

- How can we interpret this model?
- We'll repeat our exercise from a few weeks ago and plug in the two possible values of Z_i
- When $Z_i = 0$:

$$\begin{aligned}\widehat{Y}_i &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i \\ &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 \times 0 + \widehat{\beta}_3 X_i \times 0 \\ &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i\end{aligned}$$

- When $Z_i = 1$:

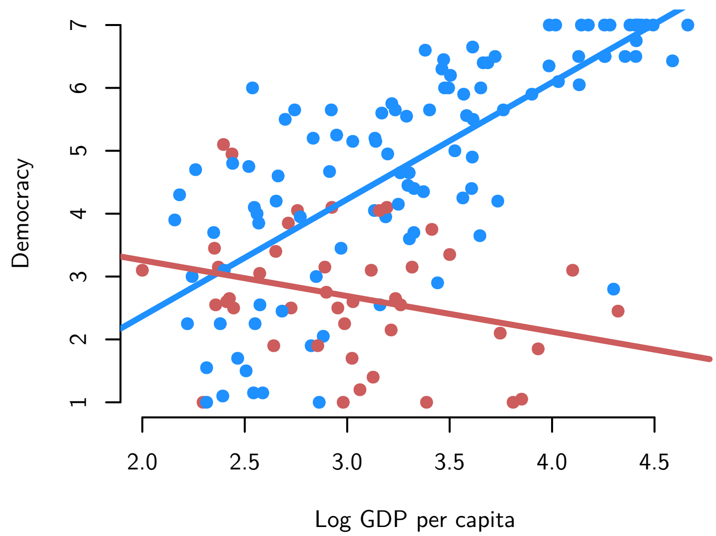
$$\begin{aligned}\widehat{Y}_i &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i \\ &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 \times 1 + \widehat{\beta}_3 X_i \times 1 \\ &= (\widehat{\beta}_0 + \widehat{\beta}_2) + (\widehat{\beta}_1 + \widehat{\beta}_3) X_i\end{aligned}$$

Example interpretation of the coefficients

- Let's review what we've seen so far:

	Intercept for X_i	Slope for X_i
Non-Muslim country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
Muslim country ($Z_i = 1$)	$\widehat{\beta}_0 + \widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_3$

```
plot(FishData$income, FishData$fhrev, ylab = "Democracy", xlab = "Log GDP per capita",
    pch = 19, bty = "n", col = ifelse(FishData$muslim == 1, "indianred", "dodgerblue"))
abline(a = coef(mod.int)[1], b = coef(mod.int)[2], col = "dodgerblue", lwd = 3)
abline(a = coef(mod.int)[1] + coef(mod.int)[3], b = coef(mod.int)[2] + coef(mod.int)[4],
    col = "indianred", lwd = 3)
```



General interpretation of the coefficients

- $\widehat{\beta}_0$: average value of Y_i when both X_i and Z_i are equal to 0
- $\widehat{\beta}_1$: a one-unit change in X_i is associated with a $\widehat{\beta}_1$-unit change in Y_i when $Z_i = 0$
- $\widehat{\beta}_2$: average difference in Y_i between $Z_i = 1$ group and $Z_i = 0$ group when $X_i = 0$
- $\widehat{\beta}_3$: change in the effect of X_i on Y_i between $Z_i = 1$ group and $Z_i = 0$

Lower order terms

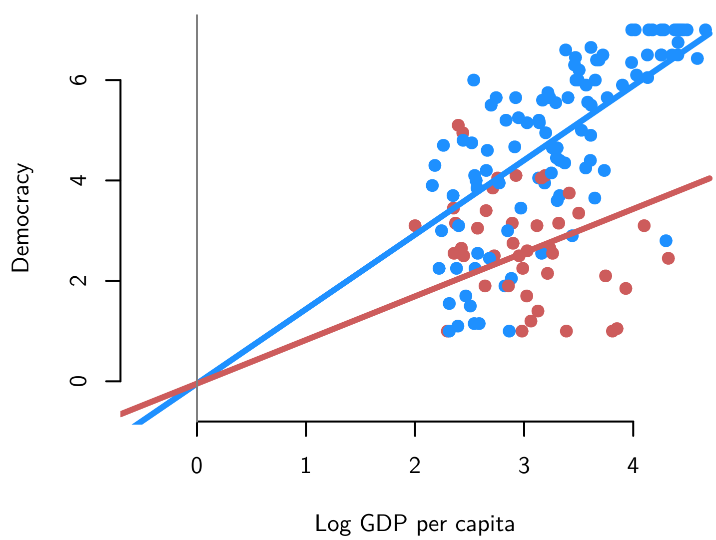
- Always include the marginal effects (sometimes called the lower order terms)
- Imagine we omitted the lower order term for `muslim`:

```
wrong.mod <- lm(fhrev ~ income + income:muslim, data = FishData)
summary(wrong.mod)
```

```
##
## Call:
## lm(formula = fhrev ~ income + income:muslim, data = FishData)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -3.5338 -0.7332  0.2524  0.8582  3.0619
```

```
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)   -0.04646    0.51333  -0.091    0.928
## income         1.48368    0.15202   9.760  < 2e-16 ***
## income:muslim -0.61372    0.07255  -8.460 2.56e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.217 on 146 degrees of freedom
## Multiple R-squared:  0.5689, Adjusted R-squared:  0.563
## F-statistic: 96.34 on 2 and 146 DF,  p-value: < 2.2e-16
```

```
plot(FishData$income, FishData$fhrev, ylab = "Democracy", xlab = "Log GDP per capita",
    pch = 19, bty = "n", col = ifelse(FishData$muslim == 1, "indianred", "dodgerblue"),
    xlim = c(-0.5, 4.5), ylim = c(-0.5, 7))
abline(a = coef(wrong.mod)[1], b = coef(wrong.mod)[2], col = "dodgerblue", lwd = 3)
abline(a = coef(wrong.mod)[1], b = coef(wrong.mod)[2] + coef(wrong.mod)[3],
    col = "indianred", lwd = 3)
abline(v = 0, col = "grey50")
```



- What's the problem here? We've restricted the intercepts to be the same for both models:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + 0 \times Z_i + \widehat{\beta}_3 X_i Z_i$$

	Intercept for X_i	Slope for X_i
Non-Muslim country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
Muslim country ($Z_i = 1$)	$\widehat{\beta}_0 + 0$	$\widehat{\beta}_1 + \widehat{\beta}_3$

- Basically, dropping the lower order term implies that there is no difference between Muslims and non-Muslims when income is 0
- Or, practically, that the intercept is the same for the two groups, but the slopes differ. Distorts slope estimates.
- Very rarely justified.

Interaction between two continuous variables

- Now let Z_i be continuous
- Z_i is the percent growth in GDP per capita from 1975 to 1998
- Is the effect of economic development for rapidly developing countries higher or lower than for stagnant economies?
- We can still define the interaction:

$$income_i \times growth_i$$

- And include it in the regression:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

Example of continuous interaction

```
mod.cont <- lm(fhrev ~ income * growth, data = FishData)
summary(mod.cont)
```

```
##
## Call:
## lm(formula = fhrev ~ income * growth, data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
```

```
## -5.0018 -0.9356  0.2241  0.9604  2.8338
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -0.1066     0.6225  -0.171   0.8643
## income          1.2922     0.1941   6.659 5.33e-10 ***
## growth         -0.6172     0.2383  -2.590   0.0106 *
## income:growth   0.2395     0.0753   3.180   0.0018 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.4 on 145 degrees of freedom
## Multiple R-squared:  0.4332, Adjusted R-squared:  0.4215
## F-statistic: 36.95 on 3 and 145 DF,  p-value: < 2.2e-16
```

```r
head(model.matrix(mod.cont))
```

```
##   (Intercept)   income growth income:growth
## 1           1 2.925312   -0.8    -2.3402497
## 2           1 3.214314    0.2     0.6428628
## 3           1 2.824126   -1.6    -4.5186013
## 4           1 3.762078    0.6     2.2572469
## 5           1 3.187803   -6.6   -21.0394974
## 6           1 4.435542    2.2     9.7581919
```

Interpretation

- With a continuous Z_i, we can have more than two values that it can take on:

	Intercept for X_i	Slope for X_i
$Z_i = 0$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
$Z_i = 0.5$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 0.5$	$\widehat{\beta}_1 + \widehat{\beta}_3 \times 0.5$
$Z_i = 1$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 1$	$\widehat{\beta}_1 + \widehat{\beta}_3 \times 1$
$Z_i = 5$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 5$	$\widehat{\beta}_1 + \widehat{\beta}_3 \times 5$

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i when $Z_i = 0$.

- The coefficient $\widehat{\beta}_2$ measures how the predicted outcome varies in Z_i when $X_i = 0$

- The coefficient $\widehat{\beta}_3$ is the change in the effect of X_i given a one-unit change in Z_i:
$$\frac{\partial \mathbb{E}[Y_i|X_i, Z_i]}{\partial X_i} = \beta_1 + \beta_3 Z_i$$

- The coefficient $\widehat{\beta}_3$ is the change in the effect of Z_i given a one-unit change in X_i:
$$\frac{\partial \mathbb{E}[Y_i|X_i, Z_i]}{\partial Z_i} = \beta_2 + \beta_3 X_i$$

Hypothesis tests

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- Due to sampling variation, the two groups will never have the **exact** same slope.
- But how do we asses if the differences in the slopes are "big enough" for us to say that the effect varies by group?
- We can test whether or not the effects for the two groups are different by testing the null hypothesis $H_0 : \beta_3 = 0$

$$\frac{\widehat{\beta}_3}{\widehat{SE}[\widehat{\beta}_3]}$$

Standard errors for marginal effects

- What if we want to get a standard error for the effect of X_i at some level of Z_i?

- We already saw that $\widehat{\beta}_1$ is the effect when $Z_i = 0$. What about other values of Z_i?

- To calculate the sampling variances (and thus the SEs), we need to use the properties of variances. Here is the expression
$$\begin{aligned}\mathbb{V}\left(\frac{\partial \mathbb{E}[Y_i|X_i, Z_i]}{\partial X_i}\right) &= \mathbb{V}(\widehat{\beta}_1 + Z_i\widehat{\beta}_3) \\ &= \mathbb{V}[\widehat{\beta}_1] + Z_i^2\mathbb{V}[\widehat{\beta}_3] + 2Z_i\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_3]\end{aligned}$$

- The variances here are the usual variances and the $\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_3]$ is the covariance between the estimator of the two coefficients (we'll learn more about this soon).

- Let's calculate the SE for the effect of income for a Muslim country. We can use the `vcov()` function to get the variances and covariances (more on this in the next few weeks):

```
## SE of effect of income at muslime = 1
var.inter <- vcov(mod.int)["income", "income"] + 1^2 * vcov(mod.int)["income:muslim",
    "income:muslim"] + 2 * 1 * vcov(mod.int)["income", "income:muslim"]
sqrt(var.inter)
```

```
## [1] 0.3277283
```

```
## SE when muslim = 0
sqrt(vcov(mod.cont)["income", "income"])
```

```
## [1] 0.1940696
```

Recentering for interaction terms

- A trick for getting R to calculate the standard errors for you is to recenter the variable so that 0 corresponds to the value you want to estimate.
- So if we wanted to estimate the effect of being a Muslim country with the associated SEs, we could use $1 - Z_i$ in place of Z_i:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2(1 - Z_i) + \beta_3 X_i(1 - Z_i) + u_i$$

- Now, $\widehat{\beta}_1$ is the slope on X_i when $1 - Z_i = 0$, or, rearranging, when $Z_i = 1$.
- We "tricked" R into calculating the standard errors for us:

```
summary(lm(fhrev ~ income * I(1 - muslim), data = FishData))
```

```
##
## Call:
## lm(formula = fhrev ~ income * I(1 - muslim), data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.8460 -0.5705  0.0940  0.8517  2.6307
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
```

```
## (Intercept)             4.3924     0.9969   4.406 2.03e-05 ***
## income                 -0.5675     0.3277  -1.732   0.0855 .
## I(1 - muslim)          -5.7413     1.1338  -5.064 1.23e-06 ***
## income:I(1 - muslim)   2.4267     0.3642   6.662 5.23e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.125 on 145 degrees of freedom
## Multiple R-squared:  0.6337, Adjusted R-squared:  0.6261
## F-statistic: 83.61 on 3 and 145 DF,  p-value: < 2.2e-16
```

- Notice that the SE is the same as we calculated before.

TESTS OF MULTIPLE HYPOTHESES

Review of t-tests

- Null hypothesis:
$$H_0 : \beta_k = 0$$
- Alternative hypothesis:
$$H_A : \beta_k \neq 0$$
- Test statistic (t-statistic):
$$t = \frac{\widehat{\beta}_k}{\widehat{SE}[\widehat{\beta}_k]}$$
- Has a $N(0, 1)$ distribution in large samples (under Assumptions 1-5) and a $t_{n-(k+1)}$ distribution under Assumptions 1-6 (when errors are conditionally Normal)

Joint null hypotheses

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + \beta_3 X_i Z_i$$

$$H_0 : \beta_1 = 0 \text{ and } \beta_3 = 0$$

$$H_A : \beta_1 \neq 0 \text{ or } \beta_3 \neq 0$$

- How can we test this null hypothesis?
- We will compare the predictive power of the model under the null and the model under the alternative

Restricted versus unrestricted models

- Unrestricted model (alternative is true):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + \beta_3 X_i Z_i$$

- Estimates:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- SSR from unrestricted model:

$$SSR_u = \sum_{i=1}^{n} (Y_i - \widehat{Y}_i)^2$$

- Restricted model (null is true):

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \beta_2 Z_i + \beta_3 X_i Z_i \\ &= \beta_0 + 0 \times X_i + \beta_2 Z_i + 0 \times X_i Z_i \\ Y_i &= \beta_0 + \beta_2 Z_i \end{aligned}$$

- Estimates:

$$\widetilde{Y}_i = \widetilde{\beta}_0 + \widetilde{\beta}_1 Z_i$$

- SSR from restricted model model:

$$SSR_r = \sum_{i=1}^{n} (Y_i - \widetilde{Y}_i)^2$$

- If the null is true, then SSR_r and SSR_u should only be different due to sampling variation.
- The bigger the reduction in the prediction errors between SSR_r and SSR_u, the less plausible is the null hypothesis.

F statistic

$$F = \frac{(SSR_r - SSR_u)/q}{SSR_u/(n-k-1)}$$

- $(SSR_r - SSR_u)$: the increase in the variation in the residuals when we remove those βs
- q = number of restrictions (numerator degrees of freedom)

- $n - k - 1$: denominator/unrestricted degrees of freedom
- Intuition:

$$\frac{\text{increase in prediction error}}{\text{original prediction error}}$$

- Each of these is scaled by the degrees of freedom

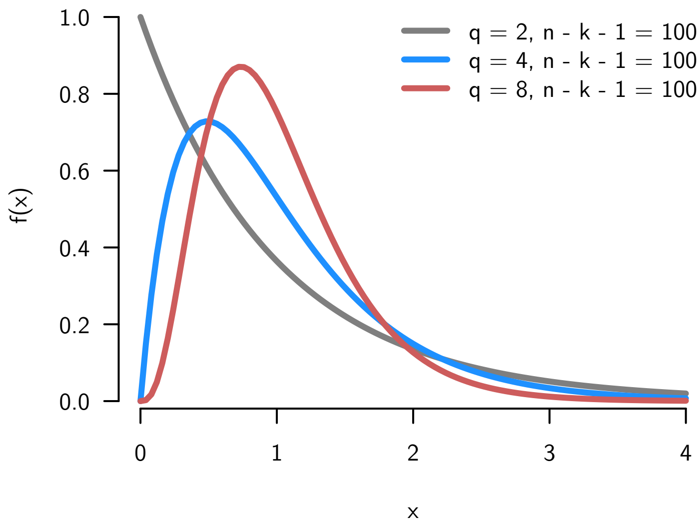
F statistic in R

```
ur.mod <- lm(fhrev ~ income * growth, data = FishData)
r.mod <- lm(fhrev ~ growth, data = FishData)
anova(r.mod, ur.mod)
```

```
## Analysis of Variance Table
##
## Model 1: fhrev ~ growth
## Model 2: fhrev ~ income * growth
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1    147 452.13
## 2    145 284.09  2    168.04 42.885 2.337e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

F distribution

```
curve(df(x, 2, 100), xlim = c(0, 4), lwd = 3, col = "grey50", bty = "n", las = 1,
    ylab = "f(x)", xlab = "x")
curve(df(x, 4, 100), xlim = c(0, 4), lwd = 3, col = "dodgerblue", add = TRUE)
curve(df(x, 8, 100), xlim = c(0, 4), lwd = 3, col = "indianred", add = TRUE)
legend("topright", legend = c("q = 2, n - k - 1 = 100", "q = 4, n - k - 1 = 100",
    "q = 8, n - k - 1 = 100"), lwd = 3, col = c("grey50", "dodgerblue", "indianred"),
    bty = "n")
```



- Ratio of two χ^2 (Chi-squared) distributions

The F test

- The F test will test this null hypothesis, but what is the sampling distribution of this F statistic?
- Very similar to the t-test. We will assume either assumptions 1-5 and in large samples, or under 1-6 (including Normality).
- With these assumptions, when the null is true, then we have:

$$\frac{(SSR_r - SSR_u)/q}{SSR_u/(n-k-1)} \sim F_{q,n-(k+1)}$$

- Under the null hypothesis, we know that the F statistic follows an F distribution with degrees of freedom q and $n-(k+1)$.
- Thus, we can perform a test of the null hypothesis by comparing our observed test statistic to the distribution of the statistic under the null.
- The F distribution tells us how much of a relative increase in the SSR we should expect if we were to add irrelevant variables to the model.
- If our calculated F statistic is large relative to the null distribution, then this means that there is more predictive power (bigger reductions in the SSR) than we would expect by random chance.
- To conduct the test, we simply choose an α, which has the same interpretation as always: the proportion of false positives you are willing to accept.
- Then we calculate the rejection region for the test. All F-tests are **one-sided tests**. Why? Because we only want to reject when the added covariates increase

our predictive power (when the SSR goes up) and this is when the F statistic is big.

- So the rejection region is going to be the region $F > c$, such that $\mathbb{P}(F > c) = \alpha$
- We can get this from R using the `qf()` function:

```
qf(0.05, 2, 100, lower.tail = FALSE)
```

```
## [1] 3.087296
```

- We might also want to calculate p-values. These would be the probability of observing an F-statistic this large or larger given the null hypothesis is true. This is just the proportion of the distribution above the observed F-statistic.
- We can calculate this in R using the `pf()` function:

```
pf(5.2, 2, 100, lower.tail = FALSE)
```

```
## [1] 0.00710471
```

F statistic for all variables

- Often, you'll an F-statistic reported along with the regression.
- This usually tests the null hypothesis of all the coefficients except the intercept being 0.
- In that case, the restricted model is just:

$$Y_i = \beta_0 + u_i$$

- And the estimate here would just be sample mean ($\widehat{\beta}_0 = \overline{Y}$)
- The SSR_r then would just be the sampling variation in Y:

$$SSR_f = \sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

Example of F-test for all variables

```
summary(ur.mod)
```

```
##
## Call:
## lm(formula = fhrev ~ income * growth, data = FishData)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.0018 -0.9356  0.2241  0.9604  2.8338
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -0.1066     0.6225  -0.171   0.8643
## income          1.2922     0.1941   6.659 5.33e-10 ***
## growth         -0.6172     0.2383  -2.590   0.0106 *
## income:growth   0.2395     0.0753   3.180   0.0018 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.4 on 145 degrees of freedom
## Multiple R-squared:  0.4332, Adjusted R-squared:  0.4215
## F-statistic: 36.95 on 3 and 145 DF,  p-value: < 2.2e-16
```

Connection to t tests

- What about an F-test with just one coefficient equal to zero? $H_0 : \beta_1 = 0$
- We already can do this with an t-test. Is there a connection to the F-test?
- Yes, it turns out that the F-statistic for a single restriction is just the square of the t-statistic:

$$F = t^2 = \left(\frac{\widehat{\beta}_1}{\widehat{SE}[\widehat{\beta}_1]} \right)^2$$

Multiple testing

- If we test all of the coefficients separately with a t-test, then we should expect that 5% of them will be significant just due to random chance.
- Illustration: randomly draw 21 variables, and run a regression of the first variable on the rest.
- By design, no effect of any variable on any other, but when we run the regression:

```r
set.seed(2138)
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Call:
## lm(formula = noise)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.1437 -0.5522  0.0697  0.6096  1.8470
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0280393  0.1138198  -0.246  0.80605
## X2          -0.1503904  0.1121808  -1.341  0.18389
## X3           0.0791578  0.0950278   0.833  0.40736
## X4          -0.0717419  0.1045788  -0.686  0.49472
## X5           0.1720783  0.1140017   1.509  0.13518
## X6           0.0808522  0.1083414   0.746  0.45772
## X7           0.1029129  0.1141562   0.902  0.37006
## X8          -0.3210531  0.1206727  -2.661  0.00945 **
## X9          -0.0531223  0.1079834  -0.492  0.62412
## X10          0.1801045  0.1264427   1.424  0.15827
## X11          0.1663864  0.1109471   1.500  0.13768
## X12          0.0080111  0.1037663   0.077  0.93866
## X13          0.0002117  0.1037845   0.002  0.99838
## X14         -0.0659690  0.1122145  -0.588  0.55829
## X15         -0.1296539  0.1115753  -1.162  0.24872
## X16         -0.0544456  0.1251395  -0.435  0.66469
## X17          0.0043351  0.1120122   0.039  0.96923
## X18         -0.0807963  0.1098525  -0.735  0.46421
## X19         -0.0858057  0.1185529  -0.724  0.47134
## X20         -0.1860057  0.1045602  -1.779  0.07910 .
## X21          0.0021111  0.1081179   0.020  0.98447
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9992 on 79 degrees of freedom
```

```
## Multiple R-squared:  0.2009, Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.4797
```

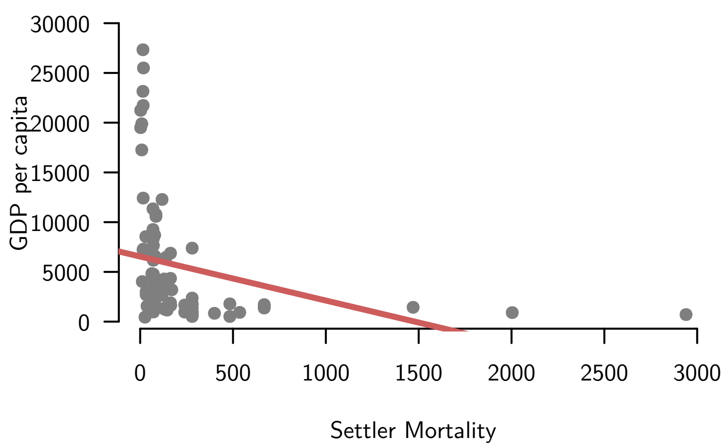
- Notice that out of 20 variables, one of the variables is significant at the 0.05 level (in fact, at the 0.01 level).
- But this is exactly what we expect: 1/20 = 0.05 of the tests are false positives at the 0.05 level
- Also note that 2/20 = 0.1 are significant at the 0.1 level. Totally expected!
- But notice the F-statistic: the variables are not *jointly* significant

NONLINEAR FUNCTIONAL FORMS

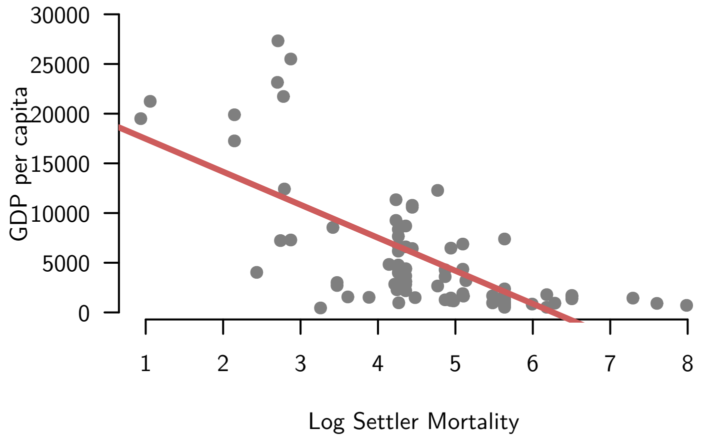
Logs of random variables

- We can account for non-linearity in X_i in a couple of ways
- One way: transform X_i or Y_i using the natural logarithm
- Useful when X_i or Y_i are positive and right-skewed
- Changes the interpretation of β_1:
  - Regress $\log(Y_i)$ on $X_i \to 100 \times \beta_1 \approx$ **percent increase** in Y_i associated with a one-unit increase in X_i
  - Regress $\log(Y_i)$ on $\log(X_i) \to \beta_1 \approx$ **percentage increase** in Y_i associated with a **one percent** increase in X_i
  - Only useful for small increments, not for discrete r.v

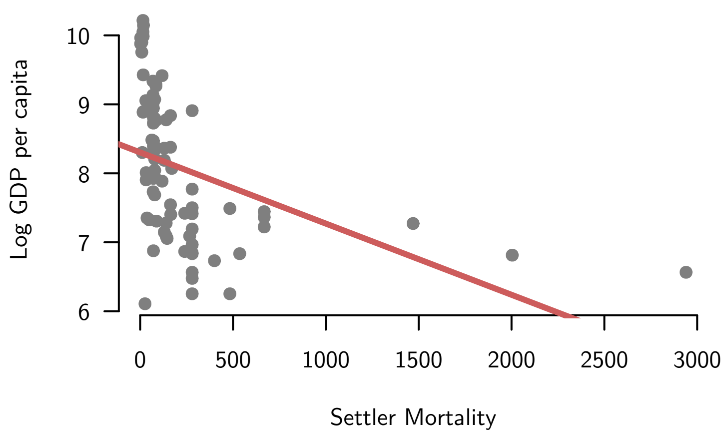
Raw scales



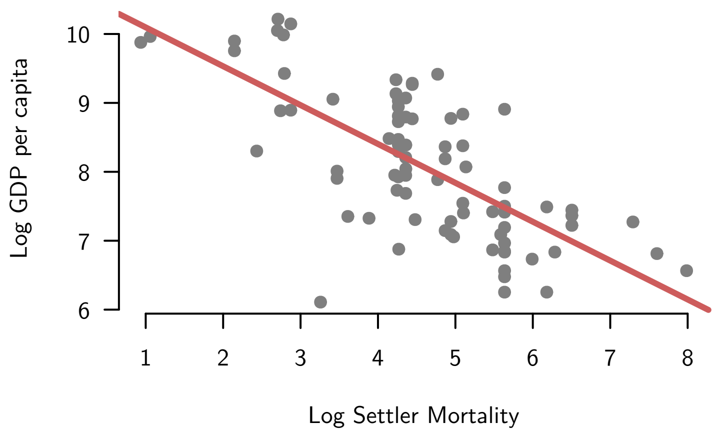
Log scale for Settler mortality



Log scale for GDP



Log scale for both



- Handy chart for interpreting logged variables:

Model	Equation	β_1 Interpretation
Level-Level	$Y = \beta_0 + \beta_1 X$	1-unit $\Delta X \rightsquigarrow \beta_1 \Delta Y$
Log-Level	$\log(Y) = \beta_0 + \beta_1 X$	1-unit $\Delta X \rightsquigarrow 100 \times \beta_1 \% \Delta Y$
Level-Log	$Y = \beta_0 + \beta_1 \log(X)$	1% $\Delta X \rightsquigarrow (\beta_1/100) \Delta Y$
Log-Log	$\log(Y) = \beta_0 + \beta_1 \log(X)$	1% $\Delta X \rightsquigarrow \beta_1 \% \Delta Y$

Adding a squared term

- Another approach: model relationship as a polynomial
- Add a polynomial of X_i to account for the non-linearity:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 X_i^2$$

- Similar to an "interaction" with itself: marginal effect of X_i varies as a function of X_i:

$$\frac{\partial \mathbb{E}[Y_i|X_i]}{\partial X_i} = \beta_1 + \beta_2 X_i$$

```
quad.mod <- lm(logpgp95 ~ raw.mort + I(raw.mort^2), data = ajr)
summary(quad.mod)
```

```
##
## Call:
## lm(formula = logpgp95 ~ raw.mort + I(raw.mort^2), data = ajr)
```

```
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.43698 -0.66321  0.00788  0.65436  1.63024
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)    8.639e+00  1.378e-01  62.687  < 2e-16 ***
## raw.mort      -3.616e-03  6.638e-04  -5.447 5.77e-07 ***
## I(raw.mort^2)  1.091e-06  2.623e-07   4.157 8.19e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.884 on 78 degrees of freedom
##   (82 observations deleted due to missingness)
## Multiple R-squared:  0.3211, Adjusted R-squared:  0.3037
## F-statistic: 18.45 on 2 and 78 DF,  p-value: 2.755e-07
```

- Plotting the results (see handout for R code):