Gov 50: 16. Random Variables

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Fall 2018

1/ Today's agenda

Where are we? Where are going?

- Learned the basics of probability.

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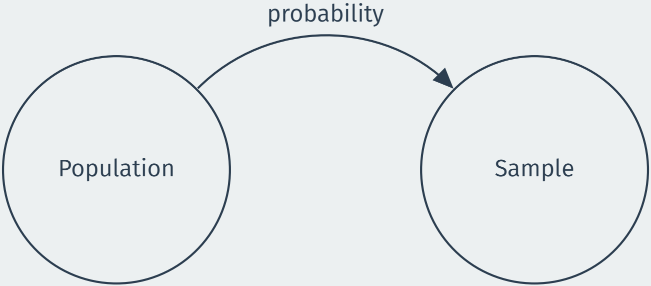
  - Addition rule
  - Conditional probability
  - Independence
- Now, random variables and probability distributions.

2/ Why probability?

Learning about populations



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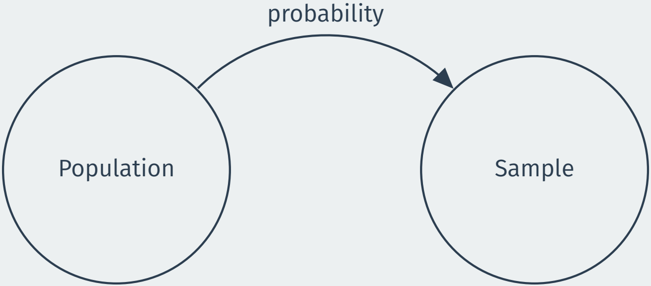

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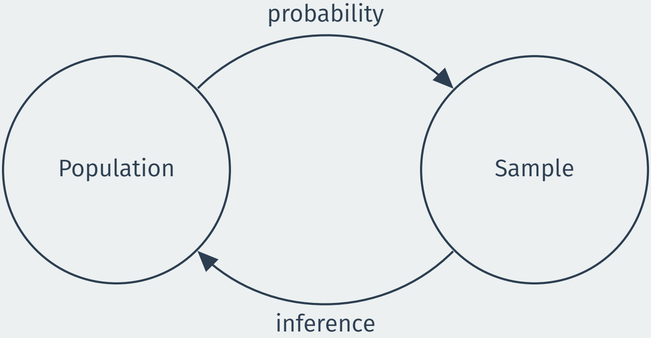

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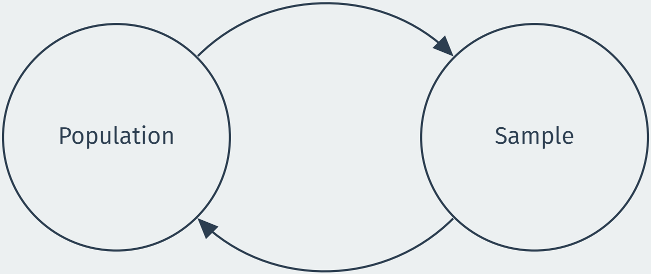

- **Probability**: formalize the uncertainty about how our data came to be.
- **Inference**: learning about the population from a set of data.

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- Probability to the rescue!

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  - Prepare 8 cups of tea, 4 milk-first, 4 tea-first
  - Present cups to friend in a **random** order
  - Ask friend to pick which 4 of the 8 were milk-first.

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- You've done your first hypothesis test and calculated your first p-value!

3/ Random variables and probabilities distributions

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- We'll think about each observation in our data frame as a r.v.

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  - $Z = 1$ if any of the 3 flips are heads.

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  - Amount of time spent on a website.

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  - $\mathbb{P}(Y > 65)$ is the share of registered voters over 65.

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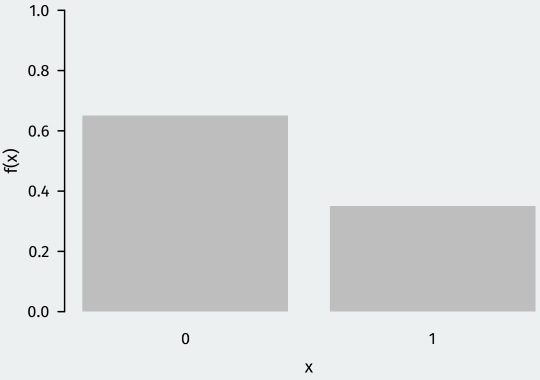

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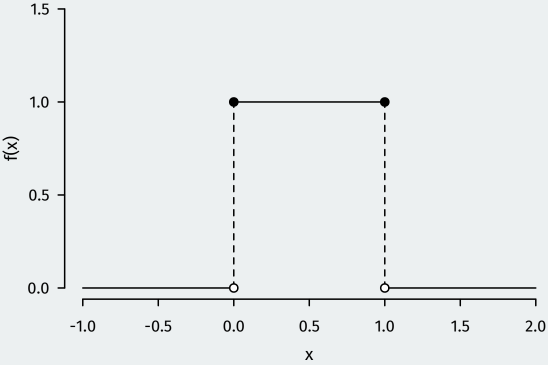

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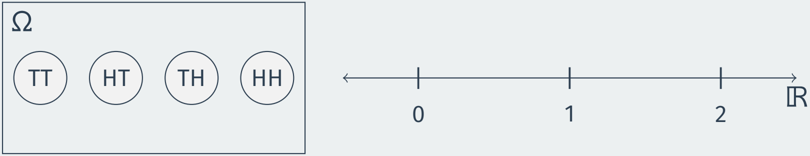

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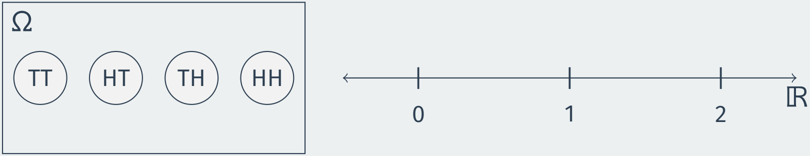

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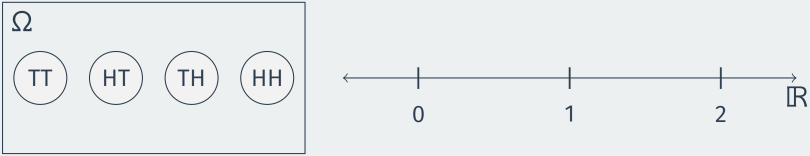

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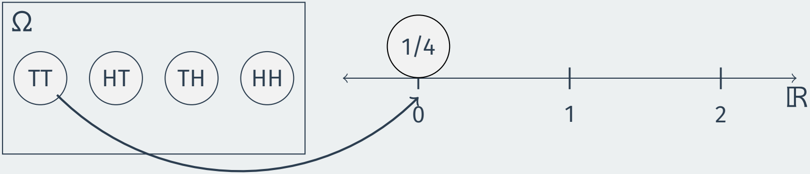

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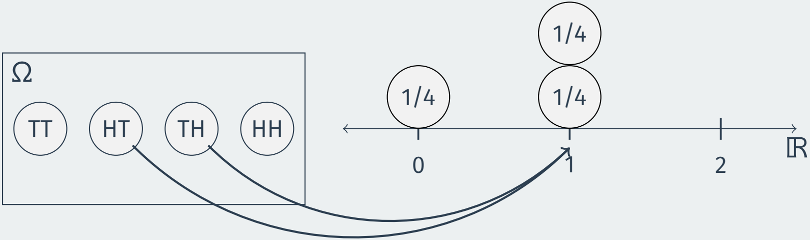

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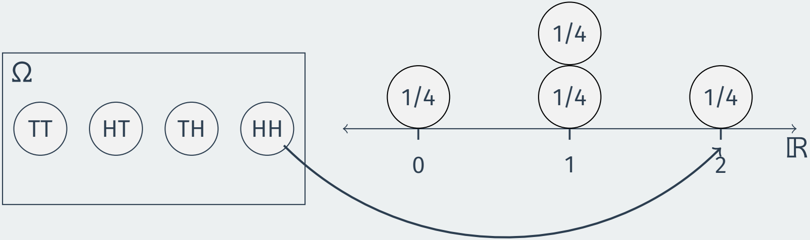

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4/ Summarizing distributions

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- With real data, we are going to try and infer these values from data on a r.v.

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- For discrete $X \in \{x_1, x_2, \dots, x_k\}$ with k levels:

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Expectation

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  - Weighted average of the values of the r.v. weighted by the probability of each value occurring.

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Let X and Y be r.v.s and a and b be constants.

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- The **standard deviation** is the (positive) square root of the variance: $\sigma_X = \sqrt{\mathbb{V}[X]}$.

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5/ Famous distributions

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  - Others in the book, but we won't focus on them.

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  - Infinite number of possible Bernoulli r.v.s: one for each value of p.

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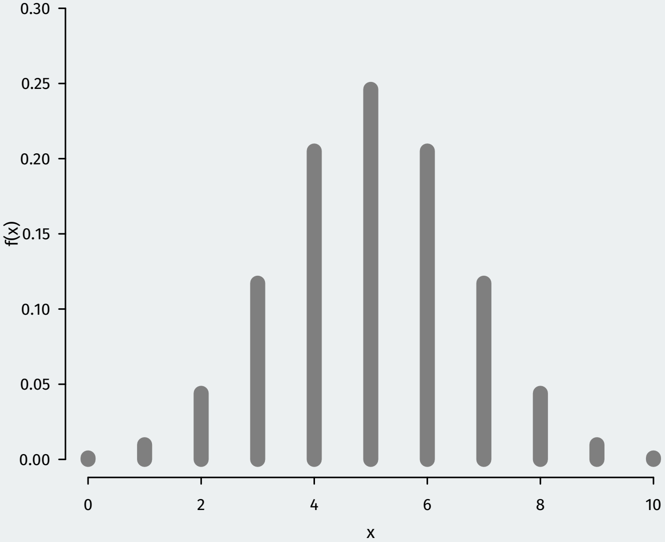
- $\rightsquigarrow \mathbb{E}[X] = np$ and $\mathbb{V}[X] = np(1 - p)$

Binomial distribution (n=10,p=0.5)



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##  [1] 7 5 3 3 4 5 7 5 5 9
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- What if drew lots of samples of size 1000? What would the distribution look like?

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- What if drew lots of samples of size 1000? What would the distribution look like?
  - $\rightsquigarrow$ what if drew a lot of samples of X?

Simulations

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draws <- rbinom(sims, size = 1000, prob = 0.42)
length(draws)
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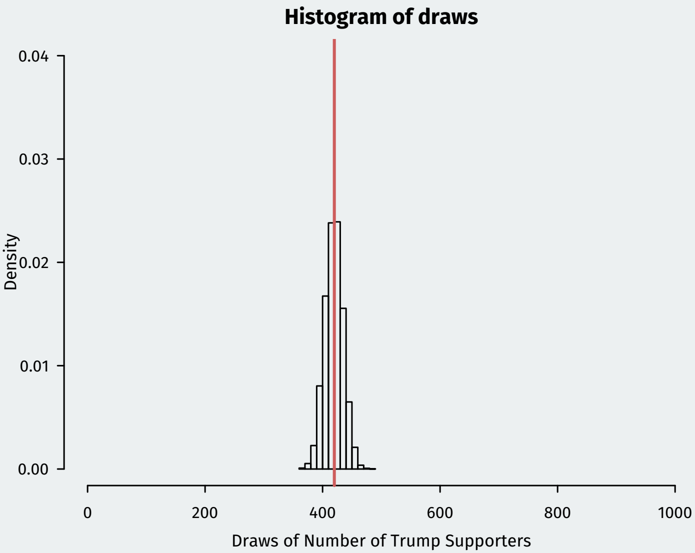
```
## [1] 0.442 0.431 0.420 0.407 0.405 0.435
```

```r
hist(draws, freq = FALSE, xlim = c(0, 1000), ylim = c(0, 0.04),
     xlab = "Draws of Number of Trump Supporters")
abline(v = 420, col = "indianred", lwd = 2)
```

Histogram of draws



Next time

- Properties of sums and means in large samples.

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- Normal distribution and the central limit theorem!