Gov 50: 15. Probability

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Fall 2018

1/ Today's agenda

Logistics

- DataCamp Assignment 5: due Thursday.

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  - See “Final Project” link on Canvas for pointers on data sets.

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  - Survey about groups due Thursday, 11/1.
  - Paragraph describing data, proposed analyses due 11/21.
  - See "Final Project" link on Canvas for pointers on data sets.
  - Feel free to email us or come to office hours to talk about ideas.

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- We need a way to talk about random variability/chance: **probability**.

2/ Probability

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- **Sample space**: Ω the set of possible outcomes.
- **Event**: any subset of outcomes in the sample space

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- An event: picking a Queen, $\{Q\clubsuit, Q\spadesuit, Q\heartsuit, Q\diamondsuit\}$

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1. (Nonnegativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Addition Rule) If two events A and B are mutually exclusive

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

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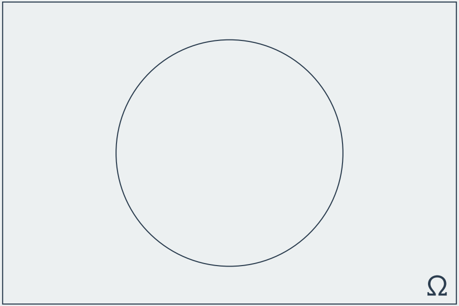

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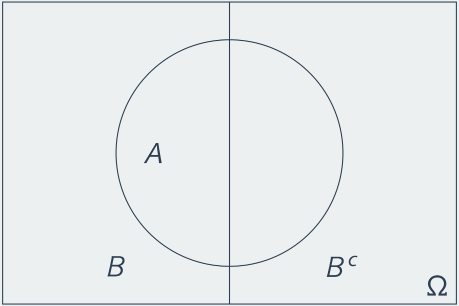

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- Conditional probability is a huge part of what we do in the empirical social sciences.

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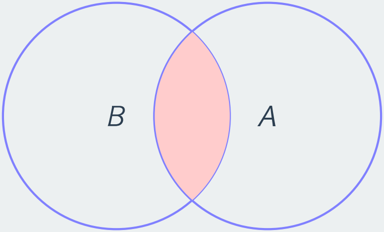
$P(A \text{ and } B)$



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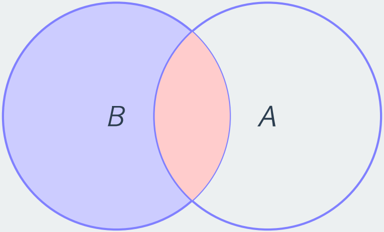
$$P(A \mid B)$$



US Senate example

	Democrats	Republicans	Independents	Total
Men	39	42	2	83
Women	12	5	0	17
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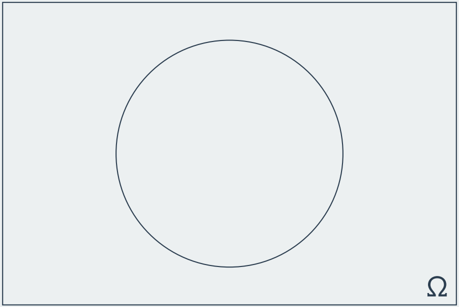
$$\mathbb{P}(A) = \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid \text{not } B)\mathbb{P}(\text{not } B)$$

Law of Total Probability



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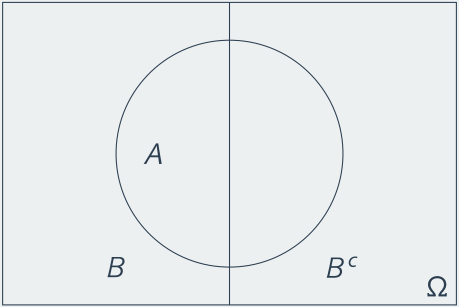

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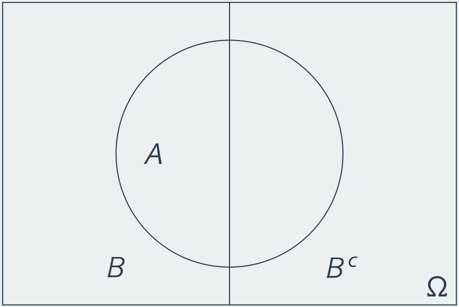

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- **Law of total probability** (with the multiplication rule):

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4/ Independence

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  - Randomly pick 2nd card from remaining 51 cards.
  - Getting an Ace in first card changes the probability of drawing an Ace for the second card.

Independence and the lottery

Every week you buy a ticket in a lottery that offers one chance in a million of winning. What is the chance that you never win, even if you keep this up for 10 years?

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- Still very small.

Wrap up

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- Next time: random variables.