Gov 50: 21. Hypothesis testing: Two-sample tests

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Fall 2018

1/ Today's agenda

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    - Final report due 12/10.

2/ Hypothesis testing review

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- Example:
  - We've learned how to estimate a causal effect from an experiment or observational study.
  - But how can we tell if the difference we estimate is real or just due to chance?
  - Hypothesis test: assume there is no effect and determine what the data would look like in that world.

Hypothesis testing procedure

Conducted with several steps:

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5. Use p-value to decide whether to reject the null hypothesis or not

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- Today: generalizing to differences in means.

3/ Two-sample tests

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- In words: does the treatment and control group have the same distribution?

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- Standard error:

$$\widehat{\text{SE}}_{\widehat{\text{ATE}}} = \sqrt{\frac{\overline{X}_T(1 - \overline{X}_T)}{n_T} + \frac{\overline{X}_C(1 - \overline{X}_C)}{n_C}} = 0.028$$

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- Using the z-transformation/standardization:

$$\frac{(\overline{X}_T - \overline{X}_C) - (\mu_T - \mu_C)}{\sqrt{\frac{\mu_T(1-\mu_T)}{n_T} + \frac{\mu_C(1-\mu_C)}{n_C}}} \sim N(0,1)$$

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- By CLT, $Z \sim N(0,1)$

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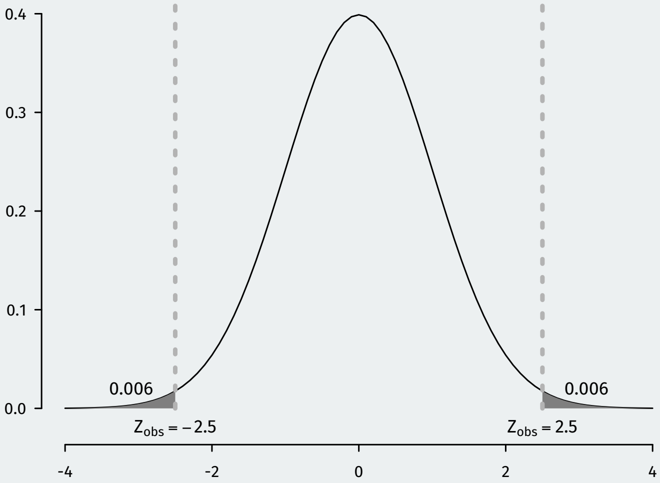
- p-value based on a two-sided test: probability of getting a difference in means this big (or bigger) if the null hypothesis were true
  - Lower p-values $\rightsquigarrow$ stronger evidence against the null.




```
2 * pnorm(2.5, lower.tail = FALSE)
```

```
## [1] 0.0124
```

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  - $\rightsquigarrow$ p-value for $H_0 : \mu_T - \mu_C = 0$ less than 0.05.
- Confidence intervals are all of the null hypotheses we **can't reject** with a test.

4/ Example: checking randomization

Checking randomization

- Load the social pressure experiment data:

```
social <- read.csv("data/social.csv")
social <- subset(social, hhsize == 2)
treated <- subset(social, messages == "Neighbors")
control <- subset(social, messages == "Control")
head(treated[,1:4])
```

```
##         sex yearofbirth primary2004  messages
## 28     male        1946           0 Neighbors
## 29   female        1932           0 Neighbors
## 80   female        1946           0 Neighbors
## 81     male        1941           0 Neighbors
## 116    male        1970           1 Neighbors
## 117  female        1971           1 Neighbors
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  - $\widehat{\sigma}_T^2$ is the sample variance of year of birth in the treated group.
  - $\widehat{\sigma}_C^2$ is the sample variance of year of birth in the control group.
- Test statistic is the same: $(\overline{X}_T - \overline{X}_C)/\widehat{\text{SE}}$

R can do the work

```
t.test(treated$yearofbirth, control$yearofbirth)
```

```
##
##  Welch Two Sample t-test
##
## data:  treated$yearofbirth and control$yearofbirth
## t = -1.26, df = 33600, p-value = 0.21
## alternative hypothesis: true difference in means is not equal to
## 95 percent confidence interval:
##  -0.292963  0.063707
## sample estimates:
## mean of x mean of y
##    1954.6    1954.7
```

5/ Power Analyses

Effect sizes

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
N of Individuals	191,243	38,218	38,204	38,218	38,201

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- **Detect** here means "reject the null of no effect"

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- If we fail to reject a null hypothesis, two possible states of the world:

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  - Probability that we reject given some specific value of the parameter $\mathbb{P}_\theta(|T| > c)$
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  - Null is false (there is a treatment effect), but test had low power.

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  - You sample 10 hiring records of each race, conduct hypothesis test and fail to reject null.
- Say to judge, "look we don't have any racial discrimination"! What's the problem?

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  - Repeat for different effect sizes.

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  - Implies $\mathbb{V}[\overline{X}_T - \overline{X}_C] = 0.2/250 + 0.2/250 = 0.0016$.
- Using these assumptions, we can derived the sampling distribution of the estimator under the proposed effect size:

$$\overline{X}_T - \overline{X}_C \approx N(0.05, 0.0016)$$

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- Can figure out the probability of this from the sampling distribution!
- Since $1.96 \times \sqrt{0.0016} = 0.078$:

$$\mathbb{P}\left(\overline{X}_T - \overline{X}_C < -0.078\right) + \mathbb{P}\left(\overline{X}_T - \overline{X}_C > 0.078\right)$$

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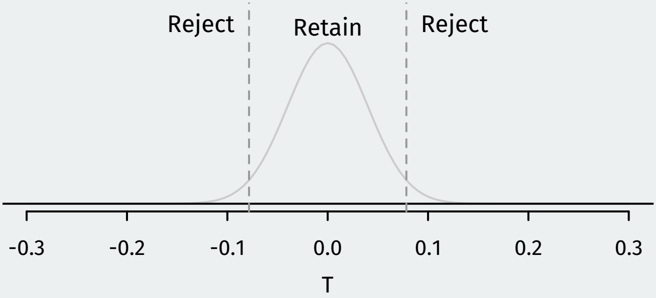
- Interpretation: if the true effect was a 5 percentage point increase in voter turnout, then we would be able to reject the null of no effect about **a quarter of the time**.

Power graph



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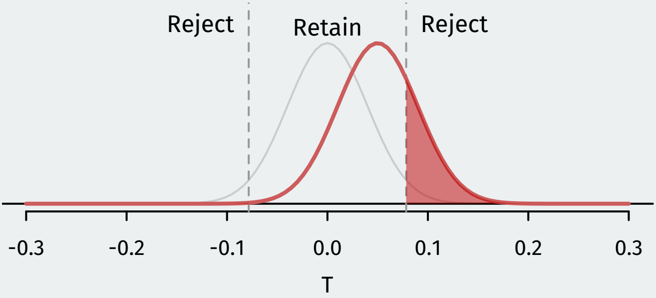

Assumed treatment effect = 0.05 and power = 0.23952.

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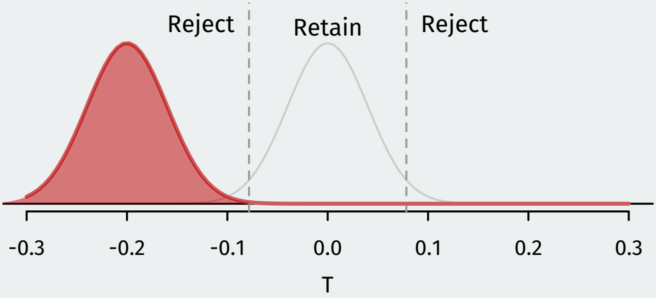
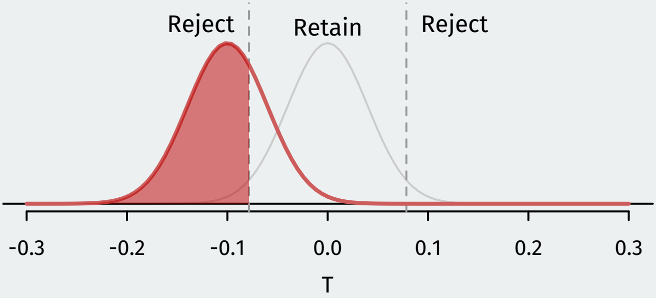


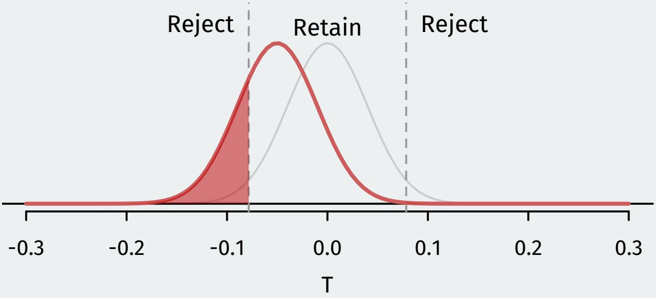
Assumed treatment effect = -0.2 and power = 0.99882.

Power graph



Assumed treatment effect = -0.1 and power = 0.70541.

Power graph



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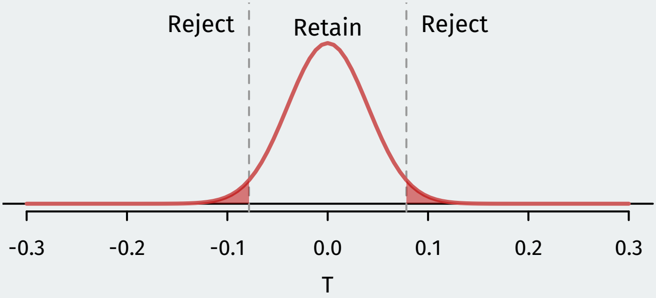

Assumed treatment effect = 0 and power = 0.05.

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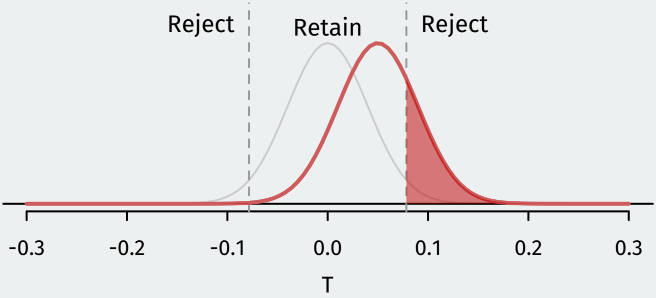

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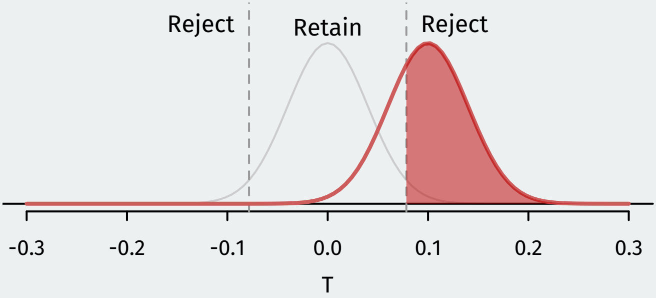

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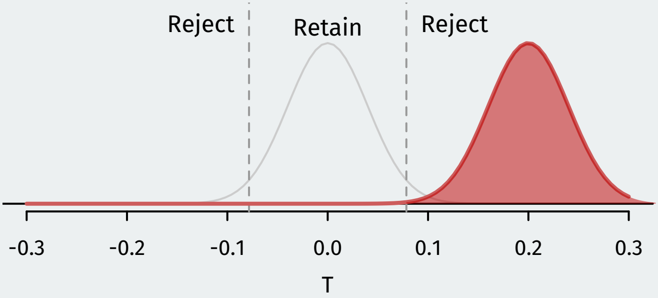

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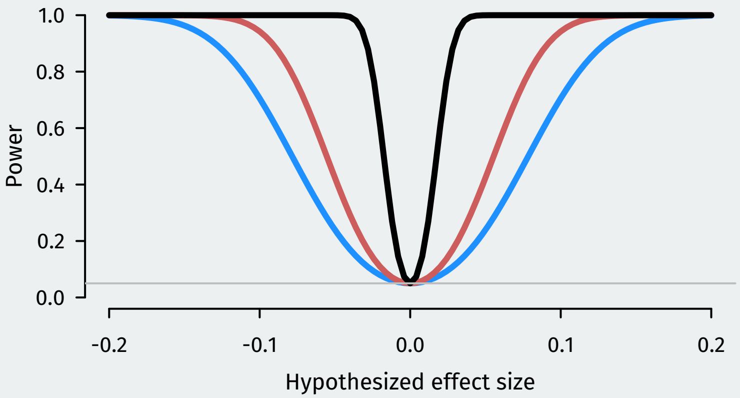

Next time

- How to conduct inference on regression coefficients.